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Relaxation Problem with a Quadratic Noise: Analysis

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The properties of a linear differential equation with an additive quadratic noise are analyzed. The graphs of the probability distribution of the process are presented for various values of the noise strength and the damping constant. The time evolution of the distribution is also shown. An infinitesimal generator of the evolution operator of the process is constructed. A diffusion-type approximation is considered and a comparison of the exact solution with the approximate solution is carried out.

KEY WORDS: Stochastic equations; nonlinear noise; master equations; exactly solved model.

1. INTRODUCTION

Stochastic differential equations are a powerful tool in the study of dynamics of systems subjected to the action of random perturbations. An evolution equation with random parameters defines a certain stochastic process x_t and the main problem is to find the probability density $P(x, t)$ of x_t with given initial conditions. In many cases the equation that governs the time evolution of the probability density is required. This leads to the concept of a master equation and, as a further consequence, to the determination of an infinitesimal generator of the evolution operator. If the evolution equation with random parameters is considered and the characteristics of random parameters are known, then, generally speaking, we know almost nothing about the process under consideration: $P(x, t)$ and(or) the master equation or(and) the generator of the process are unknown. For this purpose various approximation techniques have been proposed. On the other hand, it is worthwhile to study stochastic equations for which the whole dynamics is explicitly given.

This paper is dedicated to the memory of Prof. A. Pawlikowski.

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In this paper we analyze a nontrivial model for which these problems can be solved from end to end. The model of interest is given by the linear differential equation

$$\dot{z} = cz + b, \qquad z \in R^1 \tag{1.1}$$

with $c < 0$ and b a quadratic colored noise. Many processes in physical, chemical, and biological systems are described by Eq. (1.1). If c is fixed and b is a white noise or an Ornstein–Uhlenbeck process (a Smoluchowski one), then $P(z, t)$ can easily be obtained and has a Gaussian form. Equation (1.1) has been investigated for $b = 0$ and under the assumption that c has a white noise component[(1)] as well as that c has a nonstationary, colored noise component.[(2)] A homogeneous linear equation with the nonlinear multiplicative noise c has been considered by many authors.[(3–7),2] Pomeau[(8)] considered Eq. (1.1) for fixed c and for b a linear dichotomic Markov process. The exact time-dependent distribution function for this problem was found by Sibani and van Kampen.[(9)] The relaxation of systems under the effect of a two-level Markovian noise was also investigated by Ishii and Kitahara.[(10)]

The rest of the paper is organized as follows. In Section 2, we analyze the main properties of the exact probability distribution $P(z, t)$ for the process of interest obtained in a recent paper.[(11)] For special cases of the value of c, we give a suitable representation of $P(z, t)$ desirable for numerical evaluation. From this representation we have obtained graphs of $P(z, t)$ for different noise intensities and damping constants. Section 3 is devoted to the problem of a generator of the process. We present the integrodifferential equation with retardation satisfied by $P(z, t)$ (Section 3.1) and the master equation (Section 3.2) obtained by the elimination of the noise variable from the joint distribution. We also show that the master equation can be written in the form of a Kramers–Moyal-type expansion with the determined coefficients. The master equation can be solved by the Fourier transform and the method of characteristics and, in fact, we obtain a second method for solving the starting problem. In Section 4, the Fokker–Planck (diffusion-type) approximation is analyzed and compared with the exact solution. It permits us to verify approximation methods that have been proposed. Auxiliary mathematical formulas are given in the appendices.

[2] In Ref. 7 we obtained the exact mean value of the process using two different methods. Our methods differ from those in Refs. 5 and 6.

2. PROPERTIES OF THE PROBABILITY DISTRIBUTION

In a recent paper[11] we obtained the compact formula for the distribution $P(z, t)$ of the process z_t defined by the stochastic equation

$$\dot{z}_t = cz_t + \mu y_t^2 \tag{2.1}$$

where y_t is the colored noise

$$\langle y_t \rangle = 0, \qquad \langle y_t y_s \rangle = (\gamma/\alpha) \exp(-\alpha |t-s|) \tag{2.2}$$

and is generated by an Ornstein–Uhlenbeck process. The probability distribution has the form

$$P(z, t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega \exp[-i\omega(z - z_0 e^{ct}) + (2\alpha - c)t/4]$$
$$\times \{\tfrac{1}{2}\pi f_\omega g_\omega [J_{\nu+1}(f_\omega) Y_{\nu-1}(f_\omega(t)) - Y_{\nu+1}(f_\omega) J_{\nu-1}(f_\omega(t))]\}^{-1/2} \tag{2.3}$$

where

$$f_\omega(t) = f_\omega \exp(ct/2) \tag{2.4}$$

$$f_\omega = (4\alpha/c)(i\gamma\mu\omega/\alpha^2)^{1/2} \tag{2.5}$$

$$g_\omega = (i\gamma\mu\omega/\alpha^2)^{1/2} \tag{2.6}$$

$$\nu = 2\alpha/c \tag{2.7}$$

In (2.3), J_ν and Y_ν are the Bessel and Neumann functions, respectively.

Note that Eq. (2.1) is invariant for a change of sign of both z_t and μ. Therefore we will consider only the case $\mu > 0$. The probability density for $\mu < 0$ can be obtained from the relation

$$P(z, t; \mu) = P(-z, t; -\mu) \tag{2.8}$$

Without losing generality, we take $\mu = 1$ and for convenience we introduce a new parameter σ by the relation[12]

$$2\gamma = (\alpha\sigma)^2 \tag{2.9}$$

Then in Eqs. (2.5) and (2.6) $\gamma\mu/\alpha^2 = \sigma^2/2$ characterizes the noise strength and α characterizes the "color" of the noise y_t: if $\alpha \to \infty$, then y_t tends to the white noise ($y_t\, dt \to \sigma\, dW_t$).

In the relaxation case, $c<0$, the integrand in Eq. (2.3) can be expressed by elementary functions for the half-odd-integer order ν and then

$$c=4\alpha/(1-2n), \qquad n=1, 2, 3,... \tag{2.10}$$

In the complex plane of ω, the integrand in (2.3) is a two-valued function and has singularities on the imaginary axis. If $\mu>0$, then all singularities lie on the negative part of the imaginary axis. For example, for the case $c=-4\alpha$ and in the stationary limit $(t\to\infty)$ singularities are at the points $\omega=\omega_m$:

$$\omega_m=-i(2\pi^2/\sigma^2)(m+\tfrac{1}{2})^2 \tag{2.11}$$

for an integer m.

Let us choose the contour of integration so that it consists of the interval $(-\infty, +\infty)$ of the real axis and the semicircle lying in the lower or upper half-plane. For $z<z_0\exp(ct)$, the integration has to be carried out around the contour in the upper half-plane where the integrand is holomorphic, so $P(z, t)$ is zero. For $z>z_0\exp(ct)$ the integration has to be performed around the contour in the lower half-plane and $P(z, t)$ is different from zero. From above it follows that $P(z, t)$ is proportional to the Heaviside function $(\mu>0)$

$$P(z, t)\sim\theta(z-z_0\exp(ct)) \tag{2.12}$$

and we can determine the domain of (z, t) where $P(z, t)=0$. An explicit evaluation of $P(z, t)$ is impossible, so we have to carry out the numerical integration. Because the integrand in Eq. (2.3) is two-valued, we must specify the integration path and give a suitable representation of Eq. (2.3). It can be performed for the cases (2.10).

2.1. The Case $c=-4\alpha$ $(\nu=-1/2)$

In this case $P(z, t)$ is given by Eq. (5.3) in Ref. 12 and its integral representation desirable for the numerical calculations reads

$$P(z, t)=\frac{4f_1^{-2}(t)}{\pi\sigma^2}\int_0^\infty d\omega\,\frac{(\operatorname{sech}\sqrt{\omega})^{1/2}}{[A_1^2(\omega, t)+B_1^2(\omega, t)]^{1/4}} \times\cos\left[\frac{4\omega(z-z_0e^{-4\alpha t})}{\sigma^2f_1^2(t)}-\frac{\varphi_1(\omega, t)}{2}\right] \tag{2.13}$$

where

$$f_1(t) = 1 - e^{-2\alpha t} \tag{2.14}$$

$$A_1(\omega, t) = \cos\sqrt{\omega} + \sqrt{\omega}(\tanh\sqrt{\omega}\cos\sqrt{\omega} - \sin\sqrt{\omega})/(e^{2\alpha t} - 1) \tag{2.15}$$

$$B_1(\omega, t) = \tanh\sqrt{\omega}\sin\sqrt{\omega} + \sqrt{\omega}(\tanh\sqrt{\omega}\cos\sqrt{\omega} + \sin\sqrt{\omega})/(e^{2\alpha t} - 1) \tag{2.16}$$

and the phase $\varphi_1(\omega, t)$ is determined from the following equation:

$$\operatorname{tg}\varphi_1(\omega, t) = B_1(\omega, t)/A_1(\omega, t) \tag{2.17}$$

as an increasing function of ω. In the stationary case, Eq. (2.17) has the simple form

$$\operatorname{tg}\varphi_1(\omega) = \tanh\sqrt{\omega}\,\operatorname{tg}\sqrt{\omega} \tag{2.18}$$

The shape of the stationary distribution $P_{st}(z)$ was presented in Ref. 11. Here, Fig. 1 shows that time evolution of $P(z, t)$ obtained from Eq. (2.13). The influence of the noise intensity σ on the shape of $P(z, t)$ is very similar to that presented for $P_{st}(z)$ in Ref. 11. The influence of the initial condition on the evolution of $P(z, t)$ can be deduced from the relation (2.12).

2.2. The Case $c = -4\alpha/3$ ($\nu = -3/2$)

In this case $P(z, t)$ is transformed into

$$P(z, t) = \frac{4f_2^{-5/2}(t)}{9\pi\sigma^2}\int_0^\infty d\omega\,\frac{(2\sqrt{\omega}\,\operatorname{sech}\sqrt{\omega})^{1/2}}{[A_2^2(\omega, t) + B_2^2(\omega, t)]^{1/4}} \times \cos\left[\frac{4\omega(z - z_0 e^{-4\alpha t/3})}{9\sigma^2 f_2^2(t)} - \frac{\varphi_2(\omega, t)}{2}\right] \tag{2.19}$$

where

$$f_2(t) = 1 - e^{-2\alpha t/3} \tag{2.20}$$

$$A_2(\omega, t) = \sin\sqrt{\omega} + \tanh\sqrt{\omega}\cos\sqrt{\omega} + 2F(t)\sqrt{\omega}\cos\sqrt{\omega} + \tfrac{4}{3}F^2(t)\,\omega(\tanh\sqrt{\omega}\cos\sqrt{\omega} - \sin\sqrt{\omega}) \tag{2.21}$$

$$B_2(\omega, t) = \sin\sqrt{\omega} - \tanh\sqrt{\omega}\cos\sqrt{\omega} + 2F(t)\sqrt{\omega}\tanh\sqrt{\omega}\sin\sqrt{\omega} + \tfrac{4}{3}F^2(t)\,\omega(\tanh\sqrt{\omega}\cos\sqrt{\omega} + \sin\sqrt{\omega}) \tag{2.22}$$

$$F(t) = 1/(e^{2\alpha t/3} - 1) \tag{2.23}$$

The phase $\varphi_2(\omega, t)$ is determined from an equation analogous to (2.17).

Results of numerical calculations are sketched in Figs. 2 and 3.

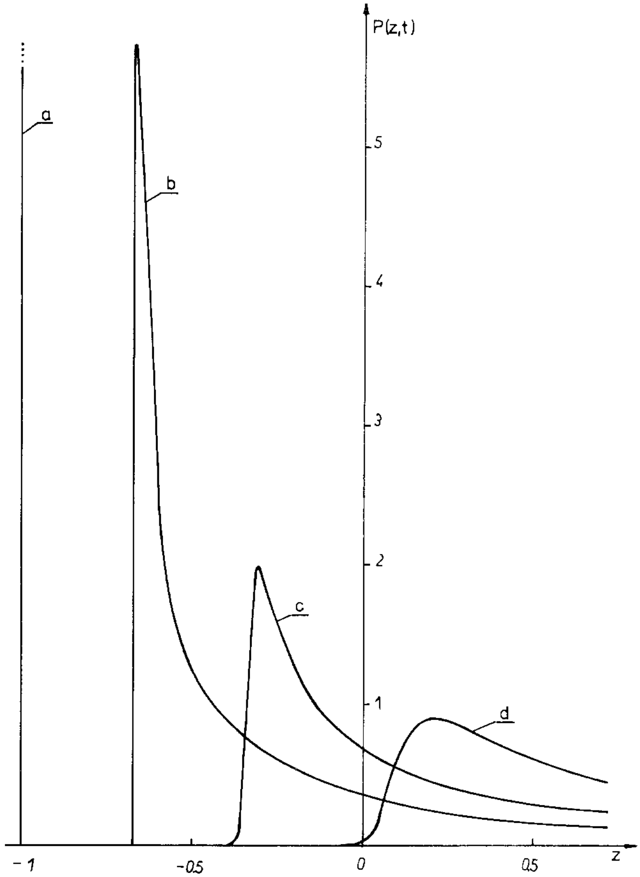

Fig. 1. Graph showing the time development of $P(z, t)$ for $\mu = 1$, $c = -4\alpha$, $\sigma^2 = 12$, and (a) the initial density, (b) $\alpha t = 0.1$, (c) $\alpha t = 0.25$, (d) $\alpha t = 1$.

2.3. The Case $c = -4\alpha/5$ ($\nu = -5/2$)

In order to show the structure of the integrand in (2.3) for different values of the damping coefficient c (or ν), we present one more case. The previous two cases refer to strong damping of the system, $|c| > \alpha$. The third case, $c = -4\alpha/5$, refers to weak damping, $|c| < \alpha$. In this case we obtain

$$P(z, t) = \frac{4 f_3^{-7/2}(t)}{25\pi\sigma^2} \int_0^\infty d\omega \frac{(4\omega\sqrt{\omega}\,\mathrm{sech}\sqrt{\omega})^{1/2}}{[A_3^2(\omega, t) + B_3^2(\omega, t)]^{1/4}} \times \cos\left[\frac{4\omega(z - z_0 e^{-4\alpha t/5})}{25\sigma^2 f_3^2(t)} - \frac{\varphi_3(\omega, t)}{2}\right] \tag{2.24}$$

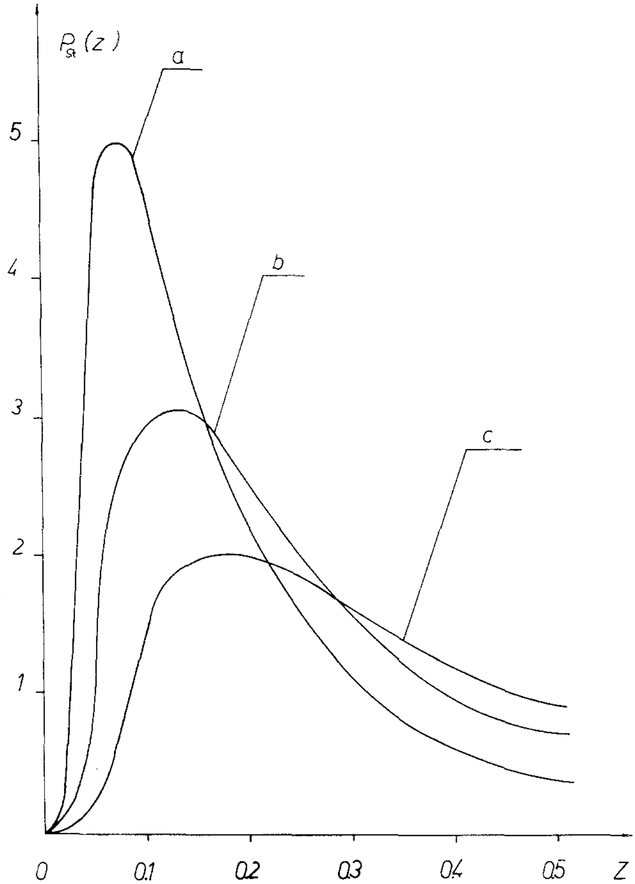

Fig. 2. Some selected examples of the stationary probability distribution $P_{st}(z)$ for $\mu = 1$, $c = -4\alpha/3$, and various values of the noise strength σ: (a) $\sigma^2 = 5/9$, (b) $\sigma^2 = 8/9$, (c) $\sigma^2 = 4/3$.

where

$$f_3(t) = 1 - e^{-2\alpha t/5} \tag{2.25}$$

and $\varphi_3(\omega, t)$ is determined as previously. Expressions for $A_3(\omega, t)$ and $B_3(\omega, t)$ are given in Appendix A.

From Eqs. (2.13), (2.19), and (2.24) one can infer the structure of the integrand (2.3) for the next values of $\nu = -7/2, -9/2, \ldots$. In Fig. 4 we present the difference of the shape of $P_{st}(z)$ for different values of the damping coefficient c. Because the dependence of $P(z, t)$ on parameters c, α, and σ is smooth, from Figs. 1–4 one can construct a full picture of the

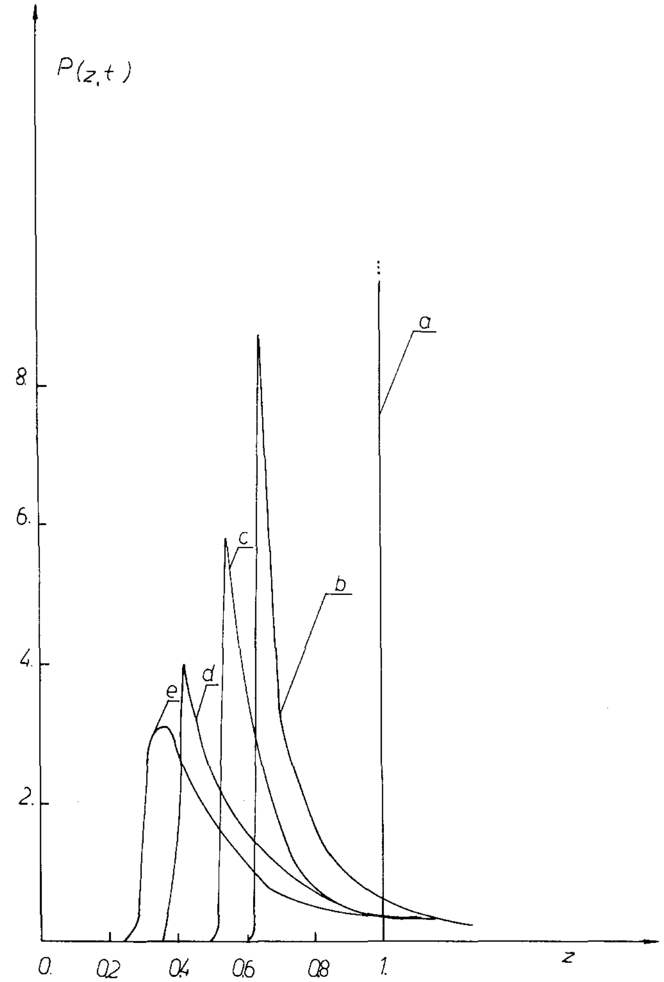

Fig. 3. Plots of $P(z, t)$ against z for $\mu = 1$, $c = -4\alpha/3$, $\sigma^2 = 4/3$, and (a) $\alpha t = 0$, (b) $\alpha t = 0.35$, (c) $\alpha t = 0.5$, (d) $\alpha t = 0.75$, (e) $\alpha t = 1$.

properties of the distribution probability $P(z, t)$. As a final (rather technical) remark, let us mention that the numerical calculation has been carried out by the Simpson method. Because $P(z, t)$, given by Eqs. (2.13), (2.19), and (2.24), is expressed as the sine and cosine Fourier transforms over a real, positive half-axis, we have verified our results using a more subtle method based on approximate expressions for the Fourier transforms.[13] In the cases considered, both methods give results which coincide within errors of the same range.

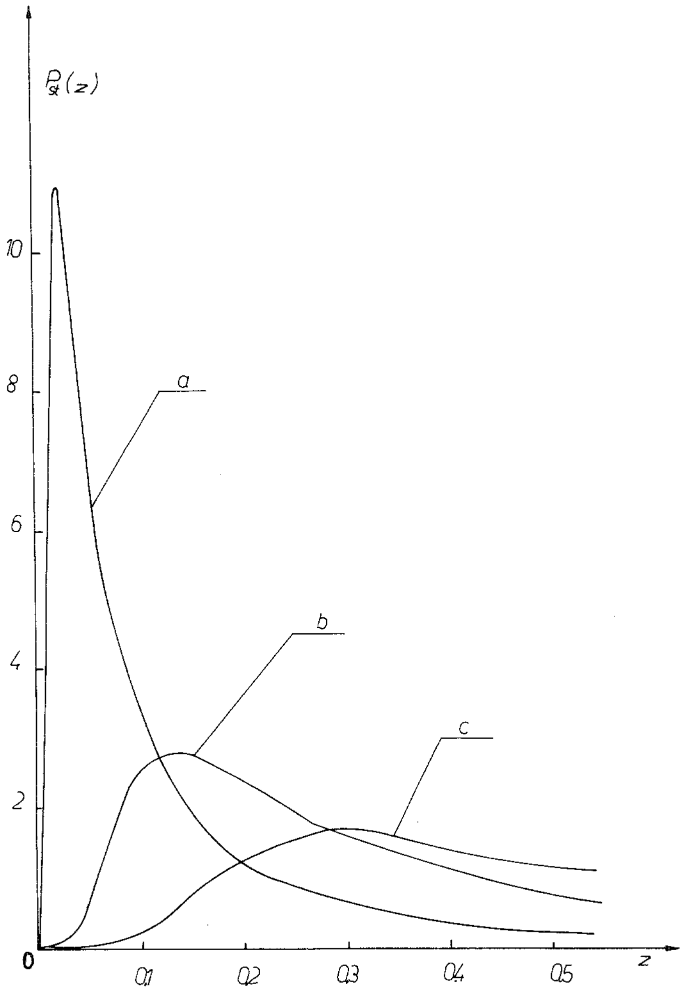

Fig. 4. Plots of $P_{st}(z)$ versus z for $\mu = 1$, $\sigma^2 = 1$, and various values of the damping parameter c: (a) $c = -4\alpha$, (b) $c = -4\alpha/3$, (c) $c = -4\alpha/5$.

3. GENERATOR OF THE PROCESS

Equation (2.3) defines the time evolution operator $U(t)$ by the relation

$$P(z, t) = U(t)\, P(z, 0) \tag{3.1}$$

with the Cauchy boundary condition

$$P(z, 0) = \delta(z - z_0) \tag{3.2}$$

The abtract theory of evolution operators does not provide much information about the family of infinitesimal generators of these operators.[14] Roughly speaking, the question about the generator L of $U(t)$ is connected with the question about a functional (differential, integrodifferential, and so on) equation satisfied by the probability distribution $P(z, t)$, (2.3). In other words, we would like to construct an evolution equation for $P(z, t)$ in the form

$$\frac{\partial}{\partial t} P(z, t) = LP(z, t) \tag{3.3}$$

The right-hand side of (3.3) should precisely be written as $(LP)(z, t)$.

3.1. The Delay Integrodifferential Equation

Because Eq. (2.1) is a particular case of an equation studied by Wódkiewicz,[15] we can write the evolution equation, which, from mathematical point of view, is the introdifferential equation with bounded delay and has the form

$$\frac{\partial}{\partial t} P(z, t) = -\frac{\partial}{\partial z}\left(cz + \frac{\mu\gamma}{\alpha}\right) P(z, t) + \int_0^t K(t-\tau)\, P(z, \tau)\, d\tau \tag{3.4}$$

where $K(t)$ is the inverse Laplace transform of $M(s)$,

$$M(s) = \int_0^\infty dt\, e^{-st}\, K(t) \tag{3.5}$$

and $M(s)$ is the operator-continued fraction[15]

$$M(s) = A \frac{1}{s - Q_2 - A \dfrac{1}{s - Q_4 - A \dfrac{1}{s - Q_6 - A \cdots} R_6} R_4} R_2 \tag{3.6}$$

where

$$A = \mu \frac{\partial}{\partial z} \tag{3.7}$$

$$Q_n = -\frac{\partial}{\partial z}\left[cz + \mu(2n+1)\frac{\gamma}{\alpha}\right] - n\alpha \tag{3.8}$$

$$R_n = \mu n(n-1)\frac{\gamma^2}{\alpha^2}\frac{\partial}{\partial z} \tag{3.9}$$

are the differential operators acting in the proper space of distributions.

Equation (3.4) is formal and useless for our aims: we are not able to determine explicitly the action of $K(t)$ on $P(z, t)$ and perform efficiently the approximate calculations. We will derive another form of the evolution equation using an elegant method based on the Martin–Rose–Siggia equations.[16]

3.2. Elimination of the Noise Variable

Let us start from the Fokker–Planck equation for the joint density $\rho(z, y, t)$ of the diffusion process (z_t, y_t),

$$\frac{\partial}{\partial t} \rho(z, y, t) = \mathscr{L} \rho(z, y, t) \tag{3.10}$$

with the infinitesimal generator

$$\mathscr{L} = -\hat{z}(cz + \mu y^2) + \alpha \hat{y} y + \gamma \hat{y}^2 \tag{3.11}$$

where, for convenience,

$$\hat{z} = \partial/\partial z, \qquad \hat{y} = \partial/\partial y \tag{3.12}$$

and

$$\rho(z, y, 0) = P(z, 0)\, P_1(y) \tag{3.13}$$

is the initial density with[12]

$$P_1(y) = (\alpha/2\pi\gamma)^{1/2} \exp(-\alpha y^2/2\gamma) \tag{3.14}$$

The reduced probability distribution $P(z, t)$ can be obtained from

$$P(z, t) = \int_{-\infty}^{+\infty} dy\, e^{t\mathscr{L}} \rho(z, y, 0) = U(t)\, P(z, 0) \tag{3.15}$$

The time derivative of $U(t)$ reads

$$\begin{aligned} \dot{U}(t) &= \int_{-\infty}^{+\infty} dy\, \mathscr{L} e^{t\mathscr{L}} P_1(y) \\ &= -c\hat{z}zU(t) - \mu\hat{z}V(t) \end{aligned} \tag{3.16}$$

where

$$V(t) = \int_{-\infty}^{+\infty} dy\, y^2 e^{t\mathscr{L}} P_1(y) \tag{3.17}$$

The generator L can be obtained from the relation

$$\dot{U}(t) = LU(t) \tag{3.18}$$

and according to Eq. (3.16) will be determined if $V(t)$ in (3.17) is expressed by $U(t)$. For this goal we adopt the method used by Haake[17] to the problem of elimination of the momentum variable from the Kramers equation.

Equation (3.17) can be rewritten in the form

$$V(t) = \int_{-\infty}^{+\infty} dy\, ye^{t\mathscr{L}}\, y(t)\, P_1(y) \tag{3.19}$$

where

$$y(t) = e^{-t\mathscr{L}}\, ye^{t\mathscr{L}} \tag{3.20}$$

For simplicity, we consider the case $c = -4\alpha$, although other cases can be treated exactly in the same manner. Using Eq. (B.9), we can write

$$V(t) = \int_{-\infty}^{+\infty} dy\, ye^{t\mathscr{L}}[\,p(\hat{z}, t) + (\alpha/\gamma)\, q(\hat{z}, t)]\; yP_1(y) \tag{3.21}$$

where the relation

$$\hat{y}P_1(y) = -(\alpha/\gamma)\; yP_1(y) \tag{3.22}$$

has been utilized. The operators $p(\hat{z}, t)$ and $q(\hat{z}, t)$ are given by Eqs. (B.12) and (B.13).

In the next step, we shift $p(\hat{z}, t)$, $q(\hat{z}, t)$, and y in (3.21) to the left of the operator $\exp(t\mathscr{L})$ to obtain

$$\begin{aligned} V(t) = {} & [\,p(\hat{z}(-t), t) + (\alpha/\gamma)\, q(\hat{z}(-t), t)] \\ & \times [\,p(\hat{z}, -t)\, V(t) + q(\hat{z}, -t)\, U(t)] \end{aligned} \tag{3.23}$$

and $\hat{z}(t)$ is given by Eq. (B.11).

In Appendix C, we show that Eq. (3.23) leads to

$$\begin{aligned} & \{\cosh[A(\hat{z})\, f(t)] + e^{-2\alpha t} A(\hat{z}) \sinh[A(\hat{z})\, f(t)]\}\; V(t) \\ & = \frac{\gamma}{\alpha} \left\{ e^{-2\alpha t} \cosh[A(\hat{z})\, f(t)] + \frac{\sinh[A(\hat{z})\, f(t)]}{A(\hat{z})} \right\} U(t) \end{aligned} \tag{3.24}$$

where $f(t) = f_1(t)$ [Eq. (2.14)] and $A(\hat{z})$ is given by Eq. (B.16). To find the inverse of the operator

$$\hat{X} = \cosh[A(\hat{z})\, f(t)] + e^{-2\alpha t} A(\hat{z}) \sinh[A(\hat{z})\, f(t)] \tag{3.25}$$

we use the Fourier transformation method for solving a linear differential equation. The action of $\hat{X}^{-1}$ on an element $g(z)$ of the distribution space is given by

$$\hat{X}^{-1}\, g(z) = \int_{-\infty}^{+\infty} dx\, W(z-x, t)\, g(x) \tag{3.26}$$

where

$$W(z, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \frac{e^{-i\omega z}}{\cos[A(i\omega) f(t)] - e^{-2\alpha t} A(i\omega) \sin[A(i\omega) f(t)]} \tag{3.27}$$

From Eq. (3.16), it follows that

$$\frac{\partial}{\partial t} P(z, t) = -c\hat{z}zP(z, t) - \mu\hat{z}V(t)\, P(z, 0) \tag{3.28}$$

Using Eqs. (3.24) and (3.26) and properties of the Fourier transforms, one find that Eq. (3.28) becomes ($c = -4\alpha$, $\mu = 1$)

$$\frac{\partial}{\partial t} P(z, t) = 4\alpha \frac{\partial}{\partial z} zP(z, t) - \frac{\alpha\sigma^2}{2} \frac{\partial}{\partial z} \int_{-\infty}^{+\infty} dx\, \mathbb{B}(z-x, t)\, P(x, t) \tag{3.29}$$

where $\mathbb{B}(z, t)$ is the Fourier transform

$$\mathbb{B}(z, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega\, e^{-i\omega z}\, \tilde{\mathbb{B}}(\omega, t) \tag{3.30}$$

of the function

$$\tilde{\mathbb{B}}(\omega, t) = \frac{\sin[A(i\omega) f(t)] + e^{-2\alpha t} A(i\omega) \cos[A(i\omega) f(t)]}{A(i\omega)\{\cos[A(i\omega) f(t)] - e^{-2\alpha t} A(i\omega) \sin[A(i\omega) f(t)]\}} \tag{3.31}$$

Equation (3.29) is the desired evolution equation for the case $c = -4\alpha$. The evolution equation for other cases can easily be derived from Eq. (2.3), assuming that the first term in the right-hand side in (3.28) should occur. It allows us to eliminate the dependence on z_0 in the time derivative of $P(z, t)$. We do not present this equation because it has the same structure as Eq. (3.29) and all the main properties of the equation can be discussed on the example of Eq. (3.29).

Equation (3.29) has a quite different structure than Eq. (3.4). While Eq. (3.4) has the structure characteristic for "non-Markovian"-type equations

$$\frac{\partial}{\partial t} P(z, t) \sim \int_0^t (\cdots)\, P(z, \tau)\, d\tau$$

Eq. (3.29) has the structure characteristic for "master"-type equations:

$$\frac{\partial}{\partial t} P(z, t) \sim \int_{-\infty}^{+\infty} [\cdots] \, P(x, t) \, dx$$

Equation (3.29) is useful for a few reasons:

(i) All expressions in (3.29) are well-determined.

(ii) Equation (3.29) can be solved explicitly.

(iii) The generator L can be given explicitly.

Let us consider point (ii). From (3.29) it follows that the characteristic function $C(\omega, t)$ obeys the following partial equation

$$\frac{\partial}{\partial t} C(\omega, t) = -4\alpha\omega \frac{\partial}{\partial \omega} C(\omega, t) + i \frac{\alpha\sigma^2}{2} \omega \tilde{\mathbb{B}}(\omega, t) \, C(\omega, t) \qquad (3.32)$$

This equation can easily be solved by the usual method of characteristics. The solution of (3.32) with the initial condition $C(\omega, t=0) = \exp(i\omega z_0)$ is given by Eq. (5.1) in Ref. 12.

Let us consider point (iii). Utilizing the property of the commutativity of convolution in (3.29) and introducing the shift operator

$$\hat{T}_x P(z, t) = P(z-x, t) = \exp\left(-x \frac{\partial}{\partial z}\right) P(z, t) \qquad (3.33)$$

we obtain (3.3) with

$$L = 4\alpha \frac{\partial}{\partial z} z - \frac{\alpha\sigma^2}{2} \frac{\partial}{\partial z} \int_{-\infty}^{+\infty} dx \, \mathbb{B}(x, t) \exp\left(-x \frac{\partial}{\partial z}\right) \qquad (3.34)$$

This is the next important and useful result.

Now, the equation that governs the time evolution of the probability function can be written as a Kramers–Moyal type equation[18]

$$\frac{\partial}{\partial t} P(z, t) = \sum_{n=1}^{\infty} \frac{\partial^n}{\partial z^n} K_n(z, t) \, P(z, t) \qquad (3.55)$$

where

$$K_n(z, t) = 4\alpha z \delta_{n,1} - \frac{\alpha\sigma^2 i^{n-1}}{2(n-1)!} \frac{\partial^{n-1}}{\partial \omega^{n-1}} \tilde{\mathbb{B}}(\omega, t) \bigg|_{\omega=0} \qquad (3.36)$$

The coefficient K_1 does not depend on time t and K_n, $n \geqslant 2$, does not depend on coordinate z.

4. DIFFUSION-TYPE APPROXIMATION ($c = -\alpha$)

By the diffusion-type equation we mean a second-order differential equation generated by the Fokker–Planck operator L_{FP} with defined drift and diffusion coefficients. The Fokker–Planck generator L_{FP} can be directly obtained from Eq. (3.35) by a truncation of the Kramers–Moyal-type expansion and keeping the first two terms $K_1(z)$ and $K_2(t)$ [Eq. (3.36)]. From (3.31) we calculate

$$\tilde{\mathbb{B}}(\omega, t)|_{\omega=0} = 1 \tag{4.1}$$

$$\left.\frac{\partial\tilde{\mathbb{B}}(\omega, t)}{\partial\omega}\right|_{\omega=0} = i\frac{\sigma^2}{6}(1 - e^{-6\alpha t}) \tag{4.2}$$

and the Fokker–Planck generator becomes

$$L_{\text{FP}} = -\frac{\partial}{\partial z}\left(\frac{\alpha\sigma^2}{2} - 4\alpha z\right) + \frac{1}{2}D^2(t)\frac{\partial^2}{\partial z^2} \tag{4.3}$$

where the diffusion coefficient $D(t)$ reads

$$D(t) = \sigma^2[(\alpha/6)(1 - e^{-6\alpha t})]^{1/2} \tag{4.4}$$

On the other hand, the operator (4.3) is the generator of the diffusion process x_t defined by the stochastic differential

$$dx_t = (\tfrac{1}{2}\alpha\sigma^2 - 4\alpha x_t)\,dt + D(t)\,dW_t \tag{4.5}$$

which may be treated as an approximate version of Eq. (2.1) (if this approximation is correct). Now, we will investigate the correctness of the approximation (4.3). The solution of the equation

$$\frac{\partial}{\partial t}P_0(z, t) = L_{\text{FP}}\,P_0(z, t) \tag{4.6}$$

has a Gaussian form

$$P_0(z, t) = (2\pi\varphi_t)^{-1/2}\exp[-(z - \beta_t)^2/2\varphi_t] \tag{4.7}$$

where

$$\beta_t = z_0 e^{-4\alpha t} + \tfrac{1}{8}\sigma^2(1 - e^{-4\alpha t}) \tag{4.8}$$

is the mean value of the process (4.5) and

$$\varphi_t = (\sigma^4/48)(1 - 4e^{-6\alpha t} + 3e^{-8\alpha t}) \tag{4.9}$$

describes fluctuations. It is interesting that expressions (4.8) and (4.9) have the same forms as Eqs. (6.1) and (6.2) in Ref. 12. This means that the diffusion-type approximation (4.3) does not destroy the main characteristics of the starting process (more precisely, the first two moments) for any time t and in this sense the approximation (4.3) leads to correct results.[4] But for many interesting problems one is interested in another properties of the process under consideration, such as the symmetry of the probability density, or the most probable path and its evolution (as in the problem of noise-induced stabilization or destabilization). Our model permits us to compare the exact (2.13) and approximate (4.7) solutions. The approximate generator (4.3) is obtained from the exact one (3.34) by a truncation of the shift operator (3.33) to the form

$$\hat{T}_x \cong 1 - x\,\partial/\partial z \tag{4.10}$$

What is the domain of validity of (4.10)? This is a problem for which in general a sufficient criterion is difficult to establish. Many authors maintain that (4.3) and in consequence (4.10) are valid for small fluctuations of the noise, i.e., if $\sigma^2 < 1$. In Figs. 5 and 6 we show graph of the exact and

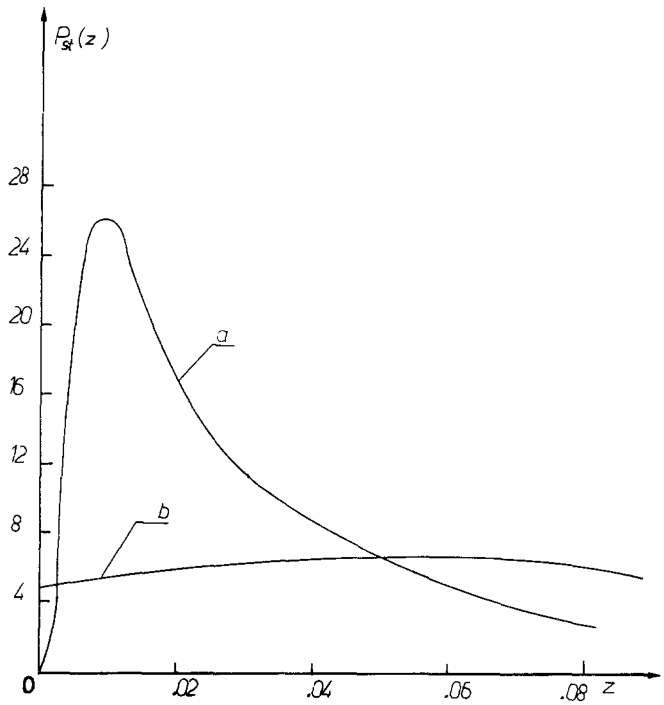

Fig. 5. Stationary density for $\mu = 1$, $c = -4\alpha$, and $\sigma^2 = 0.4$. (a) The exact distribution; (b) "approximate" distribution obtained from the diffusion-type equation (4.6).

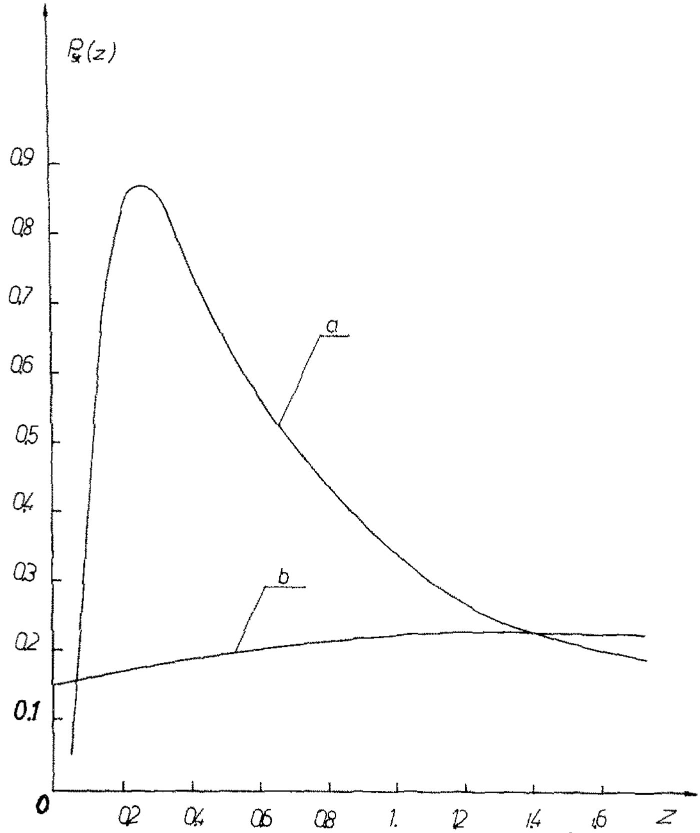

Fig. 6. Same as Fig. 5, but with $\sigma^2 = 12$.

"approximate" distribution functions $P(z, t)$ and $P_0(z, t)$, respectively, for small fluctuations, $\sigma^2 < 1$ (Fig. 5) and for large fluctuations, $\sigma^2 > 1$ (Fig. 6). A comparison of these two solutions leads to the following conclusions:

A. $P_0(z, t)$ is a symmetrical function of $z - \beta_t$, while $P(z, t)$ is asymmetrical.

B. There exists a domain of (z, t) where $P(z, t) = 0$ [see (2.12)], while $P_0(z, t)$ does not disappear for all (z, t).

C. The positions and values of the maxima of $P(z, t)$ and $P_0(z, t)$ are different.

In the above sense, the approximation (4.3) leads to unsatisfactory results.

A general conclusion from A–C is obvious: *The Fokker–Planck approximation is incorrect for the problem considered*!

5. SUMMARY

The main results can be summarized as follows (see also Refs. 11 and 12):

1. The process z_t in (2.1) is ergodic, and the probability distribution $P(z, t)$ is asymmetrical and disappears for a certain domain of (z, t).

2. The white noise limit exists for the probability distribution of the process z_t in Eq. (2.1). In this limit $P(z, t)$ coincides with the stationary density $P_{st}(z)$.

3. The master equation for $P(z, t)$ has been derived and given in analytical closed form. This equation is solvable (integrable).

4. $P(z, t)$ obeys a Kramers–Moyal-type equation with explicitly determined coefficients.

5. Two methods for solving the starting problem (2.1) have been described: the curtailed characteristic functional approach[11] and the procedure of noise variable elimination. There is a third method for solving (2.1), based on the semigroup technique.[19]

6. A comparison of the exact distribution with one obtained from the approximate diffusion-type equation has been carried out. The first two moments obtained from the averaging over $P(z, t)$ and over $P_0(z, t)$ coincide for all time and for all values of the parameters, but the other properties of $P(z, t)$ are destroyed by the diffusion-type approximation.

There is a question: Under what conditions is it possible to pass from the exact generator (3.34) to the generator (4.3) of the diffusion process (see Ref. 20 and references cited therein)? Many approximate methods are based on the expansion of interesting expressions with respect to a small parameter. In our expansion (4.10) one has no parameter at all. The problem is more subtle from the mathematical viewpoint. In fact, Eq. (3.35) is a partial differential equation with an infinite number of differential operators of any order. The situation may be similar to that in the theory of functional equations with bounded and unbounded delays.[21,14] In the case of unbounded delay, even the definition of the phase space is very complicated and has been constructed in the last decade.[21,14] For mathematicians, the better starting equation is Eq. (3.29) and it may be that an approximation technique should be applied to the kernel $\mathbb{B}(z, t)$ [Eq. (3.30)] of the integrodifferential equation (3.29).

APPENDIX A

The functions $A_3(\omega, t)$ and $B_3(\omega, t)$ in Eq. (2.24) have the form

$$A_3(\omega, t) = F_1(t)\,\omega\sqrt{\omega}\cos\sqrt{\omega} + F_2(\omega, t)\sin\sqrt{\omega} + F_3(\omega, t)\cos\sqrt{\omega}\tanh\sqrt{\omega} + 6\sqrt{\omega}\sin\sqrt{\omega}\tanh\sqrt{\omega} \tag{A.1}$$

$$B_3(\omega, t) = F_1(t)\,\omega\sqrt{\omega}\sin\sqrt{\omega}\tanh\sqrt{\omega} - F_2(\omega, t)\cos\sqrt{\omega}\tanh\sqrt{\omega} + F_3(\omega, t)\sin\sqrt{\omega} - 6\sqrt{\omega}\cos\sqrt{\omega} \tag{A.2}$$

where

$$F_1(t) = 4g(t)/5f_3^3(t) \tag{A.3}$$

$$F_2(\omega, t) = \frac{2h(t)}{5f_3^2(t)}\,\omega - \frac{4p(t)}{5f_3^4(t)}\,\omega^2 - 3 \tag{A.4}$$

$$F_3(\omega, t) = \frac{2h(t)}{5f_3^2(t)}\,\omega + \frac{4p(t)}{5f_3^4(t)}\,\omega^2 + 3 \tag{A.5}$$

and $f_3(t)$ is given by Eq. (2.25) and

$$g(t) = e^{-4\alpha t/5}(6 - e^{-2\alpha t/5}) \tag{A.6}$$

$$h(t) = e^{-2\alpha t/5}(15 - 6e^{-2\alpha t/5}) \tag{A.7}$$

$$p(t) = e^{-6\alpha t/5} \tag{A.8}$$

APPENDIX B

In Eq. (3.19) we need the quantity (3.20). Let

$$\hat{y}(t) = e^{-t\mathscr{L}}\,\hat{y}e^{t\mathscr{L}} \tag{B.1}$$

and

$$\hat{z}(t) = e^{-t\mathscr{L}}\,\hat{z}e^{t\mathscr{L}} \tag{B.2}$$

with $\mathscr{L}$ from (3.11) and $\hat{y}$ and $\hat{z}$ defined by (3.12).

The closed system of differential equations for $y(t)$, $\hat{y}(t)$, and $\hat{z}(t)$ reads

$$\dot{y}(t) = -\alpha y(t) - 2\gamma\hat{y}(t) \tag{B.3}$$

$$\dot{\hat{y}}(t) = \alpha\hat{y}(t) - 2\mu y(t)\,\hat{z}(t) \tag{B.4}$$

$$\dot{\hat{z}}(t) = -c\hat{z}(t) \tag{B.5}$$

The Martin–Rose–Siggia equations[16] (B.3)–(B.5) for our problem can be solved exactly. We have

$$\hat{z}(t) = e^{-ct}\hat{z} \tag{B.6}$$

Differentiation with respect to t of Eq. (B.3) and utilization of Eq. (B.4) with (B.6) leads to

$$\ddot{y}(t) - (4\mu\gamma\hat{z}e^{-ct} + \alpha^2)\, y(t) = 0 \tag{B.7}$$

The solution of Eq. (B.7) can be expressed by[22]

$$y(t) = Z_{2\alpha/c}\left(\frac{4i}{c}(\gamma\mu\hat{z})^{1/2}\, e^{-ct/2}\right) \tag{B.8}$$

where Z_ν stands for any solution of the Bessel equation. In the case $c = -4\alpha$ considered in Section 3.2, we obtain the following solution ($\mu = 1$):

$$y(t) = p(\hat{z}, t)\, y - q(\hat{z}, t)\, \hat{y} \tag{B.9}$$

$$\hat{y}(t) = r(\hat{z}, t)\, \hat{y} - s(\hat{z}, t)\, y \tag{B.10}$$

$$\hat{z}(t) = e^{4\alpha t}\hat{z} \tag{B.11}$$

where

$$p(\hat{z}, t) = e^{-\alpha t}\cosh[A(\hat{z})(e^{2\alpha t} - 1)] \tag{B.12}$$

$$q(\hat{z}, t) = \frac{\gamma}{\alpha}e^{-\alpha t}\frac{\sinh[A(\hat{z})(e^{2\alpha t} - 1)]}{A(\hat{z})} \tag{B.13}$$

$$r(\hat{z}, t) = e^{\alpha t}\cosh[A(\hat{z})(e^{2\alpha t} - 1)] \tag{B.14}$$

$$s(\hat{z}, t) = \frac{\alpha}{\gamma}e^{\alpha t}A(\hat{z})\sinh[A(\hat{z})(e^{2\alpha t} - 1)] \tag{B.15}$$

and

$$A^2(\hat{z}) = (\gamma/\alpha^2)\hat{z} = \tfrac{1}{2}\sigma^2\hat{z} \tag{B.16}$$

Equations (B.12)–(B.15) are in fact the well-defined power series of the differential operator $\hat{z}$.

APPENDIX C

Equation (3.23) can be rewritten in the form

$$[1 - D(\hat{z},t)p(\hat{z},-t)]V(t) = D(\hat{z},t)\; q(\hat{z},-t)\, U(t) \tag{C.1}$$

where

$$D(\hat{z},t) = p(\hat{z}(-t),t) + (\alpha/\gamma)q(\hat{z}(-t), t) \tag{C.2}$$

From (B.11)–(B.13) it follows that Eq. (C1) takes the form

$$\hat{Y}(\hat{z}, t)\ \hat{S}_1(\hat{z},\ t) = \hat{Y}(\hat{z}, t)\, \hat{S}_2(\hat{z},t) \tag{C.3}$$

where

$$\hat{Y}(\hat{z}, t) = \sinh[A(\hat{z})(1 - e^{-2\alpha t})]/A(\hat{z}) \tag{C.4}$$

$\hat{S}_1(\hat{z},\ t)$ and $S_2(\hat{z}, t)$ are the left- and right-hand sides of Eq. (3.24), respectively.

From (C.3) it follows that $\hat{S}_1(\hat{z}, t) = \hat{S}_2(\hat{z}, t)$, since the operator $\hat{Y}(\hat{z}, t)$ in (C.4) contains a term independent of the operator $\hat{z} = \partial/\partial z$. Hence Eq. (3.24) holds.

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